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AUTHORITY

AFWL ltr, 30 Nov 1971

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ELECTROMAGNETIC WAVE TRANSMISSION
THROUGH A PARABOLIC PLASMA SLAB
AT ARBITRARY ANGLES OF INCIDENCE

Whitlow W. L. Au
Lt   USAF

Clifford A. Danielson

TECHNICAL REPORT NO. AFWL-TR-66-70

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FOREWORD

This research was performed under Program Element 6.24.05.06.4, Project 5791, Task 579122.

Inclusive dates of research were October 1965 through May 1966. The report was submitted 14 June 1966 by the Air Force Weapons Laboratory Project Officer, Lt Whitlow W. L. Au (WLDE).
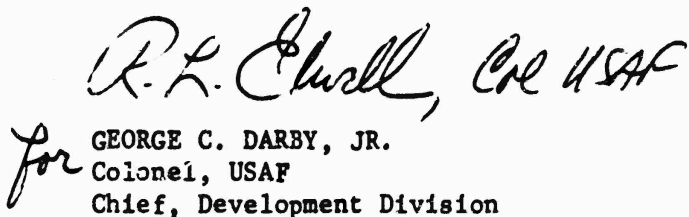
This technical report has been reviewed and is approved.

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ABSTRACT

The power transmission and reflection coefficients are derived for electro-magnetic waves propagating through a plasma slab having a parabolic electron distribution. The analysis considers transverse electromagnetic waves (TE Mode) impinging on a plasma slab at arbitrary angles of incidence. The solutions are in terms of complex hypergeometric functions and their deriva-tives. Numerical results for the transmission and reflection coefficients are plotted as functions of peak plasma frequency, peak collision frequency, signal frequency, slab thickness, and angle of incidence. The results of this study can be applied to transmission of electromagnetic energy through laboratory plasmas that are bounded by walls. Numerical results are in agreement with experimental results for a rectangular glow discharge plasma.

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CONTENTS

ILLUSTRATIONS

SYMBOLS

N_e Electron density (electrons/cm^3)

m mass of an electron

e charge of an electron

$\bar{v}$ average electron velocity

N_i density of each i constituent

Q_i electron collision cross section of the i constituent

c speed of light, 3×10^8 meters per second

σ electrical conductivity of the plasma

μ_o permeability of free space

ε_o permittivity of free space

$\varepsilon*$ effective permittivity of the plasma

n index of refraction

k_o free-space wave number, ω/c

ω signal frequency

ω_p plasma frequency

$\omega_{p_{max}}$ maximum plasma frequency

ν_c electron-neutral collision frequency

y_o center of the plasma slab

η distance from the center of the slab, $y-y_o$

θ angle of incidence

$F_x(y)$ y-dependent part of the x-component of electric field intensity in the plasma

$v(\delta)$ dependent variable of the transformed equation

δ independent variable of the transformed equation

$\bar{E}$ electric field intensity

E_I amplitude of incidence electric field

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SYMBOLS (cont'd)

E_R amplitude of reflected electric field

E_T amplitude transmitted electric field

$\overline{H}$ magnetic field intensity

$1F_1(a;c;x)$ hypergeometric function

j $\sqrt{-1}$

SECTION I

INTRODUCTION

The propagation characteristics of electromagnetic waves traversing a
bounded homogeneous plasma at various angles of incidences are well known and
relatively simple to calculate. However, in most cases (laboratory plasmas,
plasma sheaths surrounding reentry vehicles, etc.), the electron density
distribution of the plasma must be considered. Many laboratory plasmas, created
by such devices as glow discharges, R-F generators, and arc discharges, have
electron densities that can be approximated by a parabolic distribution
(references 2, 8, 9). Results of recent experiments using a plasma hollow-
cathode glow discharge tube indicate that a parabolic electron density
distribution is a good approximation to the plasma profile (reference 4).

In this present study an investigation is made of the transmission
characteristics of transverse electromagnetic waves propagating through a
plasma slab having a parabolic electron distribution. Here, the normal
incidence parabolic case considered by Bower (reference 3) is extended to
include arbitrary angles of incidence. Several errors were detected in Bower's
study which made his results for the power transmission and reflection
coefficients incorrect.

SECTION II

PLASMA PROPERTIES

In studying the interaction between electromagnetic waves and plasmas,
only the effects of the free electrons in the plasma need be considered.
The mass of the electrons is much smaller than that of the positive ions;
therefore, the electrons oscillate about their equilibrium position at a much
higher frequency than the ions. When this frequency is in the neighborhood
of the electromagnetic signal frequency, the electrons interact with the
electromagnetic wave and absorb some of the propagating energy. Therefore,
the electrical properties of the plasma, as "seen" by the electromagnetic
wave, are primarily a function of the free electrons in the plasma.

The electrical properties of a plasma slab can be described by a dielectric
constant and an electrical conductivity. These are functions of the plasma
frequency. The free electrons in a plasma are in a state of constant agitation
and each electron oscillates about its equilibrium position. The frequency
of the electron oscillation is called the plasma frequency and is defined as

$$\omega_p(y) = \left[\frac{N_e(y)e^2}{\varepsilon_o m}\right]^{1/2} \quad \text{radians/second} \quad (1)$$

where

N_e = electron density (electrons/cm^3)

ε_o = permittivity of free space

m = mass of an electron

e = charge of an electron

When the values for the various physical constants are inserted into Equation
(1), the plasma frequency is given by

$$\omega_p(y) = \left[3.18 \times 10^9 \, N_e(y)\right]^{1/2} \quad \text{radians/second} \quad (2)$$

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As the electrons oscillate, many of them will undergo collisions with neutral particles. The average number of collisions per unit time is defined as the collision frequency $\left(\nu_c\right)$. The collision frequency is a function of the electron collision cross sections of the various constituents of the plasma and their densities. The collision frequency is given by

$$\nu_c = \bar{v} \sum_i N_i Q_i \quad \text{collisions/second} \qquad (3)$$

where

$\bar{v}$ = average electron velocity

N_i = density of each i constituent

Q_i = electron collision cross section of the i constituent (m^2)

In this work, the collision frequency is assumed to be constant throughout the plasma slab. Albini and John (reference 1) have shown that the assumption of a constant collision frequency introduces very small errors when the electron density variation is large compared to the variation in collision frequency. For a reentry plasma sheath, the electron density can vary from zero at the boundaries to 10^{13} electrons/cm^3 and higher in the boundary layer. The collision frequency, however, will usually vary only by a factor of 100. In a hollow cathode glow discharge, the electron density can vary by a factor of 10^{11} across the plasma with only a small variation in the collision frequency distribution (reference 2).

The conductivity and permittivity of a plasma can be written in terms of the plasma frequency, signal frequency, and collision frequency as, respectively,

$$\sigma(y) = -j \frac{\omega_p^2(y)\epsilon_o}{\left(\omega - j\nu_c\right)} \qquad (4)$$

$$\epsilon^*(y) = \epsilon_o - j \frac{\sigma(y)}{\omega} \qquad (5)$$

Equations (4) and (5) can be separated into real and imaginary parts to give

$$\sigma(y) = \frac{\epsilon_o \nu_c \omega_p^2(y)}{\omega^2 + \nu_c^2} - j \frac{\epsilon_o \omega \omega_p^2(y)}{\omega^2 + \nu_c^2} \qquad (4a)$$

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and

$$\varepsilon^*(y) = \varepsilon_o \left[1 - \frac{\omega_p^2(y)}{\omega^2 + \nu_c^2} \right] - j \frac{\varepsilon_o \nu_c \omega_p^2(y)}{\omega(\omega^2 + \nu_c^2)} \qquad (5a)$$

SECTION III

ELECTROMAGNETIC WAVES IN A PLASMA

Maxwell's equations for a time-harmonic electromagnetic wave of radian frequency ω in an isotropic plasma are given by

$$\nabla \cdot \overline{E} = 0 \tag{6}$$

$$\nabla \cdot \overline{H} = 0 \tag{7}$$

$$\nabla \times \overline{E} = -j\omega\mu_0\overline{H} \tag{8}$$

$$\nabla \times \overline{H} = (\sigma + j\omega\epsilon)\overline{E} \tag{9}$$

The wave equation can be derived by taking the curl of Equation (8), inserting the result into Equation (9), and using Equation (6). The wave equation for the electric field in a plasma is

$$\nabla^2\overline{E} + \omega^2\mu_0\epsilon_0 \left[\frac{\epsilon}{\epsilon_0} -j \frac{\sigma(y)}{\omega\epsilon_0} \right] \overline{E} = 0 \tag{10}$$

Equation (10) can be written as

$$\nabla^2\overline{E} + k_0^2 \frac{\epsilon^*}{\epsilon_0} \overline{E} = 0 \tag{11}$$

where

$$k_0 = \omega/c$$

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

$$\frac{\epsilon^*(y)}{\epsilon_0} = \left[1 -j \frac{\sigma(y)}{\omega\epsilon_0} \right]$$

However, the index of refraction of a plasma medium is equal to the square root of its dielectric constant (reference 7). Therefore, Equation (11) becomes

$$\nabla^2\overline{E} + k_0^2 n^2(y) \overline{E} = 0 \tag{12}$$

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where

$$n(y) = \sqrt{\frac{\epsilon^\star(y)}{\epsilon_o}} = \text{index of refraction}$$

Assume that an electromagnetic wave impinges upon an inhomogeneous plasma slab as shown in figure 1. Only the TE mode, for which the electric field is perpendicular to the plane of incidence, will be considered. The angle θ_T at which the transmitted wave exits the plasma and the angle θ_R of the reflected wave will be the same as the angle θ_I of the incident wave. However, the exit point will not be in line with the incident signal since refraction occurs in the plasma (reference 7).
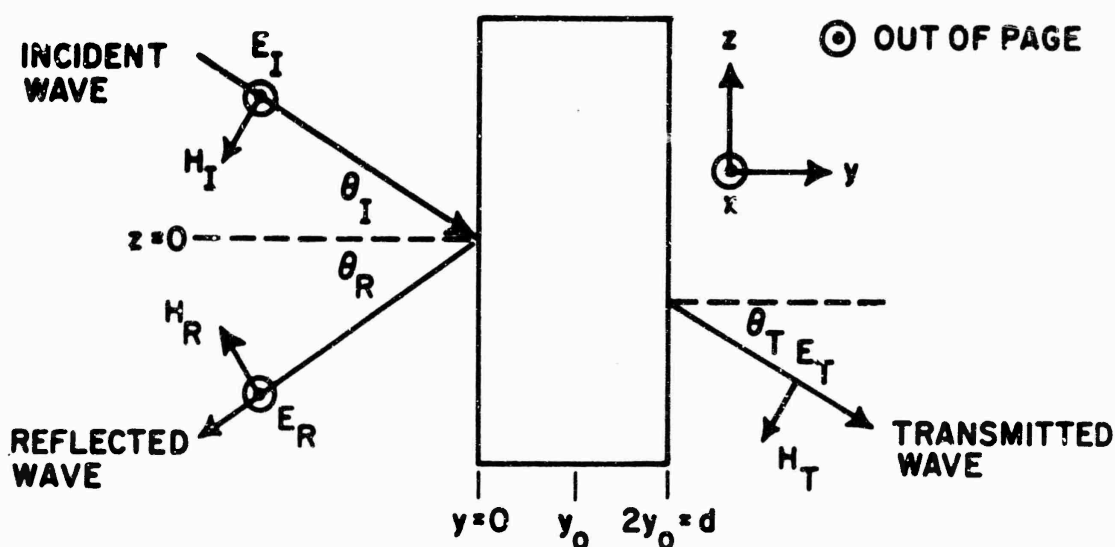

Figure 1. Geometry of Wave-Plasma Interaction

The wave equation can now be written as

$$\frac{\partial^2 E_x(y,z)}{\partial y^2} + \frac{\partial^2 E_x(y,z)}{\partial z^2} + k_o^2 n^2(y) E_x(y,z) = 0 \tag{13}$$

The Maxwell equations relating the curl of $\overline{E}$ with $\overline{H}$, Equation (8), are

$$j\omega\mu_o H_z = + \frac{\partial E_x}{\partial y} \tag{14}$$

and

$$j\omega\mu_o H_y = - \frac{\partial E_x}{\partial z} \tag{15}$$

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A solution to Equation (13) by the method of separation of variables (see Appendix I) is

$$E_x(y,z) = F_x(y)e^{jk_o z \sin\theta} \qquad (16)$$

where $F_x(y)$ satisfies the equation

$$\frac{d^2 F_x(y)}{dy^2} + k_o^2 \left[n^2(y) - \sin^2\theta \right] F_x(y) = 0 \qquad (17)$$

The effective dielectric constant can be rearranged by first dividing Equation (4a) by $\omega\varepsilon_o$

$$\frac{\sigma}{\omega\varepsilon_o} = \frac{\nu_c}{\omega} \frac{\left[\dfrac{\omega_p(y)}{\omega} \right]^2}{1 + \left[\dfrac{\nu_c}{\omega} \right]^2} - j \frac{\left[\dfrac{\omega_p(y)}{\omega} \right]^2}{1 + \left[\dfrac{\nu_c}{\omega} \right]^2} \qquad (18)$$

The index of refraction $n(y)$ can now be written as

$$n^2(y) = \frac{\varepsilon^*(y)}{\varepsilon_o} = 1 - \frac{\left[\dfrac{\omega_p(y)}{\omega} \right]^2}{1 + \left[\dfrac{\nu_c}{\omega} \right]^2} + j \frac{\nu_c}{\omega} \frac{\left[\dfrac{\omega_p(y)}{\omega} \right]^2}{1 + \left[\dfrac{\nu_c}{\omega} \right]^2} \qquad (19)$$

Let

$$u(y) = \left[\frac{\omega_p(y)}{\omega} \right]^2$$

and

$$a = \frac{1}{1 - j \dfrac{\nu_c}{\omega}} = \frac{1}{1 + \left[\dfrac{\nu_c}{\omega} \right]^2} + j \frac{\left[\dfrac{\nu_c}{\omega} \right]^2}{1 + \left[\dfrac{\nu_c}{\omega} \right]^2}$$

then

$$n^2(y) = 1 - au(y) \qquad (20)$$

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Equation (17) can now be written as

$$\frac{d^2 F_x(y)}{dy^2} + k_o^2 \left[\cos^2\theta - au(y) \right] F_x(y) = 0 \tag{21}$$

The parabolic electron distribution shown in figure 2 can be represented by the equation

$$u(y) = U_m - \frac{U_m}{y_o^2} \left(y - y_o\right)^2 \tag{22}$$

where

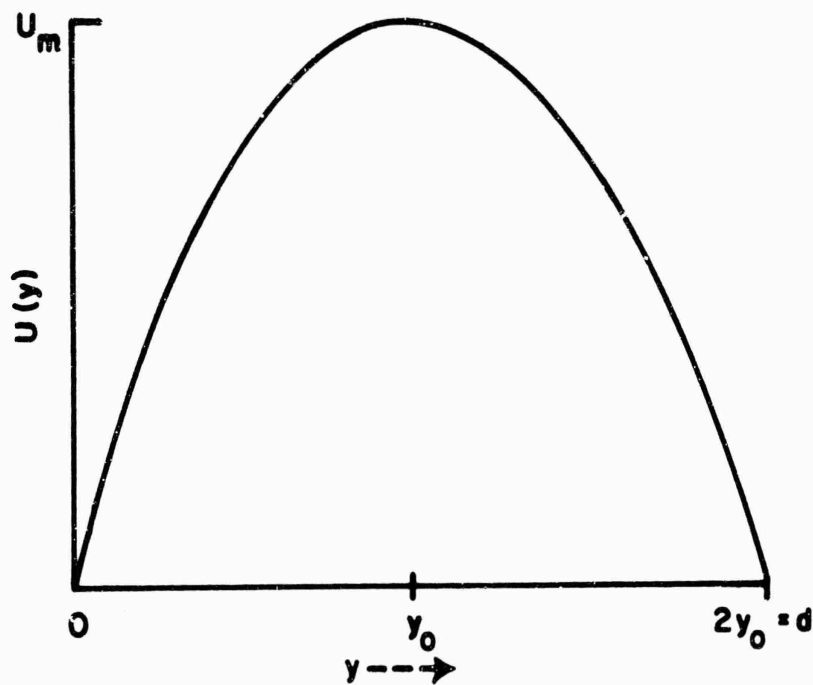$$U_m = \left[\frac{\omega_{p_{max}}}{\omega} \right]^2$$

Figure 2. Parabolic Electron Density Distribution

Now, let

$$A = k_o^2 \left(\cos^2\theta - aU_m \right)$$

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$$B = -k_o^2 \frac{aU_m}{y_o^2}$$

and

$$\eta = y - y_o$$

Equation (21) becomes

$$\frac{d^2 F_x(y)}{dy^2} + (A + B\eta^2) F_x(y) = 0 \tag{23}$$

SECTION IV

SOLUTION TO THE WAVE EQUATION

The task of this section is to solve Equation (23). which can be solved by using a power series technique or by transforming the equation into one having a known solution. In this study, Equation (23) will be transformed into another equation which has a known solution and an inverse transformation will be used to obtain the desired solution. Using the transformation

$$F_x(y) = e^{-\delta/2} v(\delta)$$

$$\delta = \jmath n^2 \sqrt{B}$$

Equation (23) can be written as (see Appendix II)

$$\delta^2 \frac{d^2 v(\delta)}{d\delta^2} + (1/2 - \delta) \frac{dv(\delta)}{d\delta} - \frac{(\sqrt{B} + \jmath A)}{4 \sqrt{B}} v(\delta) = 0 \qquad (24)$$

Letting

$$h = \frac{\sqrt{B} + \jmath A}{2 \sqrt{B}}$$

then Equation (24) becomes

$$\delta^2 \frac{d^2 v(\delta)}{d\delta^2} + (1/2 - \delta) \frac{dv(\delta)}{d\delta} - \frac{h}{2} v(\delta) = 0 \qquad (25)$$

Equation (25) is a confluent hypergeometric differential equation (reference 5) of the form

$$x^2 y'' + (c - x) - ay = 0 \qquad (26)$$

The general solution to Equation (25) is

$$v(\delta) = K_1 v_1(\delta) + K_2 v_2(\delta) \qquad (27)$$

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where

$$v_1(\delta) = {}_1F_1\left(\frac{h}{2}; \frac{1}{2}; \delta\right)$$

and

$$v_2(\delta) = \delta^{1/2}{}_1F_1\left(\frac{h}{2} + \frac{1}{2}; \frac{3}{2}; \delta\right)$$

The symbol ${}_1F_1$ is the hypergeometric function and is defined as

$${}_1F_1(a;c;x) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n x^n}{(c)_n n!} \tag{28}$$

The terminology $(a)_n$ and $(c)_n$ are factorial functions defined as

$$(A)_n = A(A+1)(A+2) \ldots (A+n-1), \text{ for } n \geq 1$$

$$(A)_o = 1 \text{ for } A \neq 0$$

and

$$(A)_o = 0 \text{ for } A = 0 \tag{29}$$

where A is any number.

By substituting Equation (27) into Equation (16) and letting

$$\beta^2 = j\sqrt{B}$$

the general solution of Equation (23) is

$$F_x(y) = e^{-\frac{\beta^2 \eta^2}{2}}\left[K_1 F_{x1}(\eta) + K_2 F_{x2}(\eta)\right] \tag{30}$$

The hypergeometric functions $F_{x1}(\eta)$ and $F_{x2}(\eta)$ and its derivatives can now be written as

$$F_{x1}(\eta) = 1 + \sum_{n=1}^{\infty} \frac{\left(\frac{h}{2}\right)_n (\beta\eta)^{2n}}{\left(\frac{1}{2}\right)_n n!} \tag{31}$$

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$$F_{x1}'(\eta) = \sum_{n=1}^{\infty} \frac{(2n)\left(\frac{h}{2}\right)_n \beta^{2n}\eta^{2n-1}}{\left(\frac{1}{2}\right)_n n!} \tag{32}$$

$$F_{x2}'(\eta) = \beta\eta \left[1 + \sum_{n=1}^{\infty} \frac{\left(\frac{h+1}{2}\right)_n \beta^{2n}\eta^{2n-1}}{\left(\frac{3}{2}\right)_n n!} \right] \tag{33}$$

$$F_{x2}'(\eta) = \beta + \sum_{n=1}^{\infty} \frac{(2n+1)\left(\frac{h+1}{2}\right) \beta^{2n+1}\eta^{2n}}{\left(\frac{3}{2}\right)_n n!} \tag{34}$$

Substituting Equation (30) into Equation (16), the electric field in the plasma is given by

$$E_x(y,z) = e^{-\frac{\beta^2\eta^2}{2}} \left[K_1 F_{x1}(\eta) + K_2 F_{x2}(\eta) \right] e^{jk_o z\sin\theta} \tag{35}$$

The magnetic fields can be obtained from Equations (14) and (15)

$$H_z = -\frac{e^{-\frac{\beta^2\eta^2}{2}}}{j\omega\mu_o} \left\{ \beta^2\eta \left[K_1 F_{x1}(\eta) + K_2 F_{x2}(\eta) \right] \right.$$
$$\left. - \left[K_1 F_{x1}'(\eta) + K_2 F_{x2}'(\eta) \right] \right\} e^{-jk_o z\sin\theta} \tag{36}$$

$$H_y = -\frac{k_o\sin\theta}{\omega\mu_o} e^{-\frac{\beta^2\eta^2}{2}} \left[K_1 F_{x1}(\eta) + K_2 F_{x2}(\eta) \right] e^{-jk_o z\sin\theta} \tag{37}$$

SECTION V

TRANSMISSION AND REFLECTION COEFFICIENTS

For a plane wave impinging on a plasma slab as in figure 1, the electric and magnetic field can be described by the following equations (reference 7):

For

$$y \leq 0$$

$$E_x(y,z) = E_1 e^{-jk_o(y \cos\theta - z \sin\theta)} + E_R e^{jk_o(y \cos\theta + z \sin\theta)} \tag{38}$$

$$H_z(y,z) = \frac{k_o\cos\theta}{\omega\mu_o} \left\{ -E_I e^{-jk_o(y \cos\theta - z \sin\theta)} \right.$$

$$\left. +E_R e^{jk_o(y \cos\theta + z \cos\theta)} \right\} \tag{39}$$

For

$$y \geq 2y_o$$

$$E_x(y,z) = E_T e^{-jk_o(y \cos\theta - z \sin\theta)} \tag{40}$$

$$H_z(y,z) = - \frac{k_o\cos\theta}{\omega\mu_o} E_T e^{jk_o(y \cos\theta - z \sin\theta)} \tag{41}$$

The boundary conditions at $y = 0$ can be applied by equating Equation (38) to Equation (35) and Equation (36) to Equation (39), and noting that at $y = 0$, $\eta = y_o$. This will give

$$E_I + E_R = e^{-\frac{\beta^2 y_o^2}{2}} \left[K_1 F_{x1}(-y_o) + K_2 F_{x2}(-y_o) \right] \tag{42}$$

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$$-E_I + E_R = \frac{e^{-\frac{\beta^2 y_o^2}{2}}}{jk_o \cos\theta} \left\{ \beta^2 y_o \left[K_1 F_{x1}(-y_o) + K_2 F_{x2}(-y_o) \right] \right.$$

$$\left. + \left[K_1 F_{x1}'(-y_o) + K_2 F_{x2}'(-y_o) \right] \right\} \tag{43}$$

The hypergeometric functions $F_{x1}(\eta)$ and $F_{x2}'(\eta)$ of Equations (31) and (34) are even functions of η. $F_{x1}'(\eta)$ and $F_{x2}(\eta)$ of Equations (32) and (33) are odd functions of η. Therefore,

$$F_{x1}(-y_o) = F_{x1}(y_o)$$

$$F_{x1}'(-y_o) = -F_{x1}'(y_o)$$

$$F_{x2}(-y_o) = -F_{x2}(y_o)$$

$$F_{x2}'(-y_o) = F_{x2}(y_o)$$

Equations (42) and (43) can now be written as

$$E_I + E_R = e^{-\frac{\beta^2 y_o^2}{2}} \left[K_1 F_{x1}(y_o) - K_2 F_{x2}(y_o) \right] \tag{44}$$

$$-E_I + E_R = \frac{e^{-\frac{\beta^2 y_o^2}{2}}}{jk_o \cos\theta} \left\{ \beta^2 y_o \left[K_1 F_{x1}(y_o) - K_2 F_{x2}(y_o) \right] \right.$$

$$\left. - \left[K_1 F_{x1}'(y_o) - K_2 F_{x2}'(y_o) \right] \right\} \tag{45}$$

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The boundary conditions at $y = 2y_o$ can be applied by equating Equation (35) to Equation (40) and Equation (36) to Equation (41), and noting that $\eta = y_o$. This will give

$$E_T \, e^{-2jk_o\cos\theta} = e^{-\frac{\beta^2 y_o^2}{2}} \left[K_1 F_{x1}(y_o) + K_2 F_{x2}(y_o) \right] \tag{46}$$

$$E_T \, e^{-2jk_o\cos\theta} = \frac{e^{-\frac{\beta^2 y_o^2}{2}}}{jk_o\cos\theta} \left\{ \beta^2 y_o \left[K_1 F_{x1}(y_o) + K_2 F_{x2}(y_o) \right] \right.$$

$$\left. - \left[K_1 F'_{x1}(y_o) + K_2 F'_{x2}(y_o) \right] \right\} \tag{47}$$

Subtracting Equations (46) and (47) gives

$$K_1 \left\{ F_{x1}(y_o) \left(1 - \frac{\beta^2 y_o}{jk_o\cos\theta} \right) + \frac{F'_{x1}(y_o)}{jk_o\cos\theta} \right\}$$

$$+ K_2 \left\{ F_{x2}(y_o) \left(1 - \frac{\beta^2 y_o}{jk_o\cos\theta} \right) + \frac{F'_{x1}(y_o)}{jk_o\cos\theta} \right\} = 0 \tag{48}$$

The ratio K_2/K_1 can be written as

$$\frac{K_2}{K_1} = \frac{F_{x1}(y_o) \left(1 - \dfrac{\beta^2 y_o}{jk_o\cos\theta} \right) + \dfrac{F'_{x1}(y_o)}{jk_o\cos\theta}}{F_{x2}(y_o) \left(\dfrac{\beta^2 y_o}{jk_o\cos\theta} - 1 \right) - \dfrac{F'_{x2}(y_o)}{jk_o\cos\theta}} \tag{49}$$

Solving for E_I and E_R from Equations (42) and (43) gives

$$E_I = e^{-\frac{\beta^2 y_o^2}{2}} \left\{ K_1 \left[F_{x1}(y_o) \left(1 - \frac{\beta^2 y_o}{jk_o \cos\theta} \right) - \frac{F_{x1}'(y_o)}{jk_o \cos\theta} \right] \right.$$

$$\left. - K_2 \left[F_{x2}(y_o) \left(1 - \frac{\beta^2 y_o}{jk_o \cos\theta} \right) + \frac{F_{x2}'(y_o)}{jk_o \cos\theta} \right] \right\} \tag{50}$$

$$E_R = e^{-\frac{\beta^2 y_o^2}{2}} \left\{ K_1 \left[F_{x1}(y_o) \left(1 + \frac{\beta^2 y_o}{jk_o \cos\theta} \right) - \frac{F_{x1}'(y_o)}{jk_o \cos\theta} \right] \right.$$

$$\left. - K_2 \left[F_{x2}(y_o) \left(1 + \frac{\beta^2 y_o}{jk_o \cos\theta} \right) - \frac{F_{x2}' y_o}{jk_o \cos\theta} \right] \right\} \tag{51}$$

Adding Equations (50) and (51) to Equation (48) gives

$$E_I = K_1 \left[F_{x1}(y_o) \left(1 - \frac{\beta^2 y_o}{jk_o \cos\theta} \right) + \frac{F_{x1}'(y_o)}{jk_o \cos\theta} \right] \tag{52}$$

$$E_R = K_1 \left[F_{x1}(y_o) - \frac{K_2}{K_1} F_{x2}(y_o) \frac{\beta^2 y_o}{jk_o \cos\theta} + \frac{K_2}{K_1} \frac{F_{x2}'(y_o)}{jk_o \cos\theta} \right] \tag{53}$$

Solving for E_T in Equations (46) and (47) gives

$$E_T = e^{j2k_o \cos\theta} \left\{ K_1 \left[F_{x1}(y_o) \left(1 + \frac{\beta^2 y_o}{jk_o \cos\theta} \right) - \frac{F_{x1}'(y_o)}{jk_o \cos\theta} \right] \right.$$

$$+ K_2 \left[F_{x2}(y_o) \left(1 + \frac{\beta^2 y_o}{jk_o \cos\theta} \right) - \frac{F_{x2}'(y_o)}{jk_o \cos\theta} \right] \right\} e^{-\frac{\beta^2 y_o^2}{2}} \tag{54}$$

Adding Equation (54) to Equation (48) gives

$$E_T = e^{j2k_o \cos\theta} K_1 \left[F_{x1}(y_o) + \frac{K_2}{K_1} F_{x2}(y_c) \right] e^{-\frac{\beta^2 y_o^2}{2}} \tag{55}$$

The power reflection and transmission coefficients are defined by, respectively,

$$T = \left| \frac{E_T}{E_I} \right|^2 \quad \text{and} \quad R = \left| \frac{E_R}{E_I} \right|^2$$

Dividing Equations (54) and (53) by Equation (52), the power transmission and reflection coefficients are given by, respectively,

$$T = \left| \frac{F_{x1}(y_o) + \frac{K_2}{K_1} F_{x2}(y_o)}{\left(1 - \frac{\beta^2 y_o}{jk_o \cos\theta} \right) F_{x1}(y_o) + \frac{1}{jk_o \cos\theta} F_{x1}'(y_o)} \right|^2 \tag{56}$$

$$R = \left| \frac{F_{x1}(y_o) - \frac{\beta^2 y_o}{jk_o \cos\theta} \frac{K_2}{K_1} \left[F_{x2}(y_o) + \frac{1}{jk_o \cos\theta} F_{x2}'(y_o) \right]}{\left(1 - \frac{\beta^2 y_o}{jk_o \cos\theta} \right) F_{x1}(y_o) + \frac{1}{jk_o \cos\theta} F_{x1}'(y_o)} \right|^2 \tag{57}$$

SECTION VI

RESULTS AND CONCLUSIONS

The power transmission and reflection coefficients were programmed for a CDC-6600 Digital Computer. The approximation of a parabolic electron density distribution for a rectangular glow discharge tube was checked and the results are shown in figure 3. The experimental points were obtained by personnel at McDonnell Aircraft Corporation (reference 4). The transmission and reflection coefficients were also calculated using a numerical technique, after the electron density distribution was obtained from Langmuir probe measurements. As can be seen in figure 3, the parabolic electron density approximation agrees very closely with the experimental results and the results obtained by the numerical method.

The power transmission and reflection coefficients are plotted as a function of signal frequency, collision frequency, slab thickness, and angle of incidence in figures 4 through 11. Some general conclusions of the electromagnetic transmission properties of a parabolic plasma slab are

a. The plasma becomes more transparent to electromagnetic signals as the signal frequency, in radian, increase above the peak plasma frequency (figure 4);

b. For a plasma with collision frequencies lower than the peak plasma frequency, the reflection coefficient changes abruptly when the signal frequency in radian equals the peak plasma frequency (figure 5). Therefore, the peak electron density of a parabolic plasma, with $\dfrac{\nu}{\omega_p(y_o)} \le 2$, can be determined from reflectometer measurements;

c. When the collision frequency is much higher than the peak plasma frequency (greater than 10), the transmission coefficient is independent of the signal frequency (figure 6);

d. For a given plasma thickness, the transmission coefficient is greater for higher signal frequency (figure 10);

e. The plasma becomes more opaque to the electromagnetic signal as the angle of signal incidence increases (figure 11);

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f.   The transmission and reflection coefficients for the parabolic electron
density case behave in a manner similar to the homogeneous plasma case, when
the electron density, collision frequency, signal frequency, thickness and
angle of incidences are varied. However, the reflection coefficient of a
homogeneous plasma is somewhat higher because of the sharp changes in the
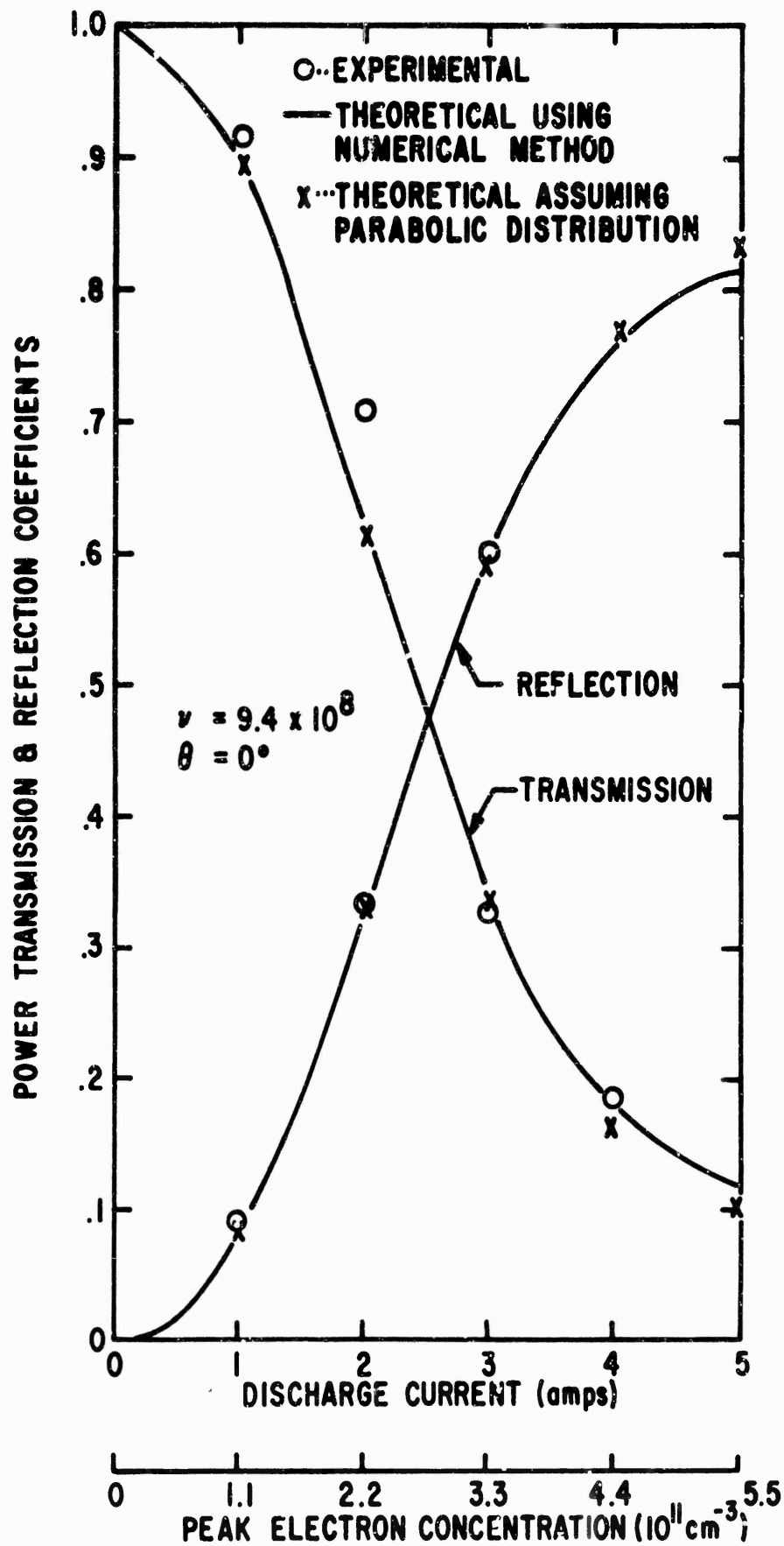boundaries.

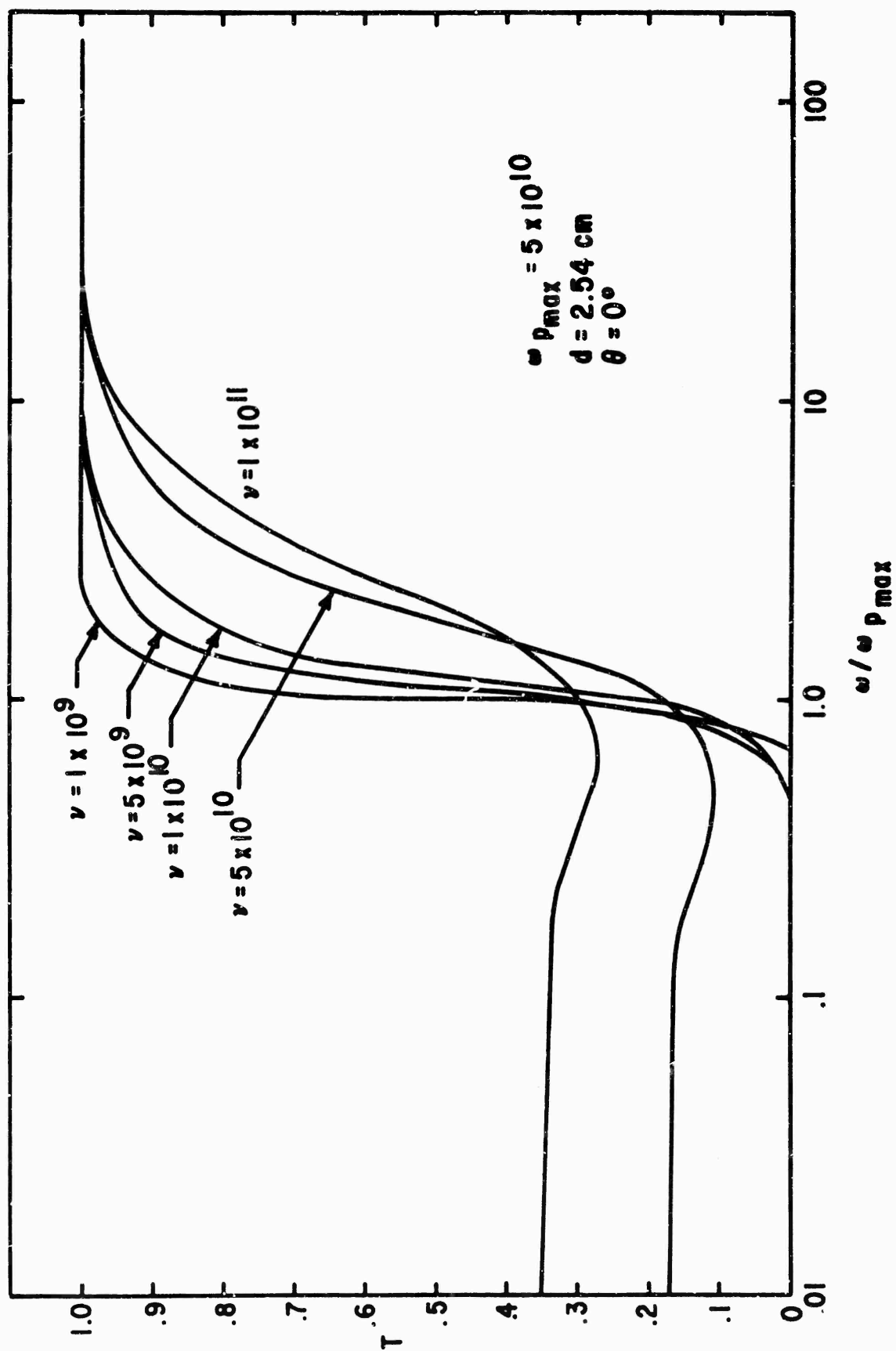Figure 3. Transmission and Reflection Coefficients for Glow Discharge Tube

Figure 4. Transmission Coefficient versus $\omega/\omega_{p_{max}}$
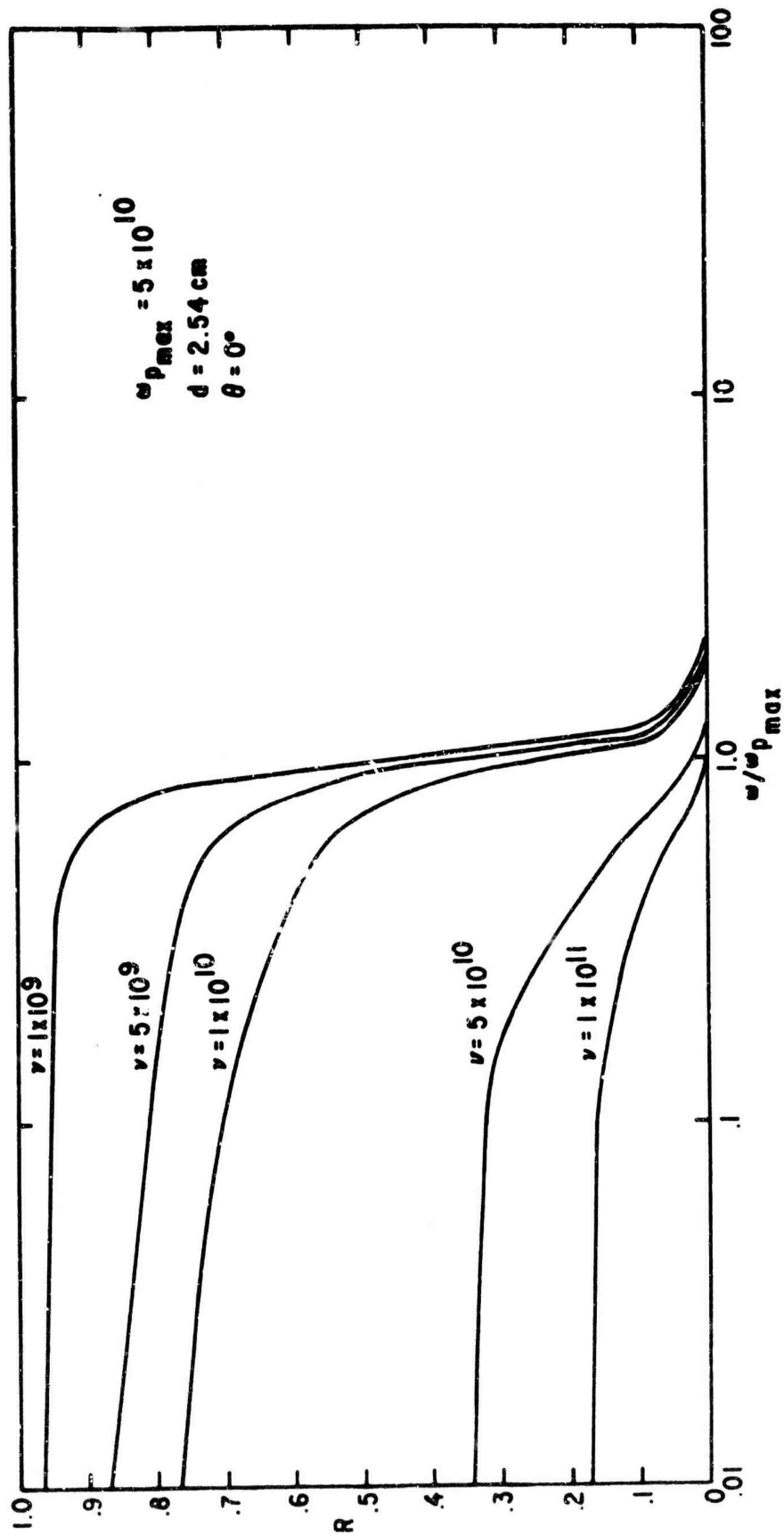
Figure 5. Reflection Coefficient versus $\omega/\omega_{p_{max}}$
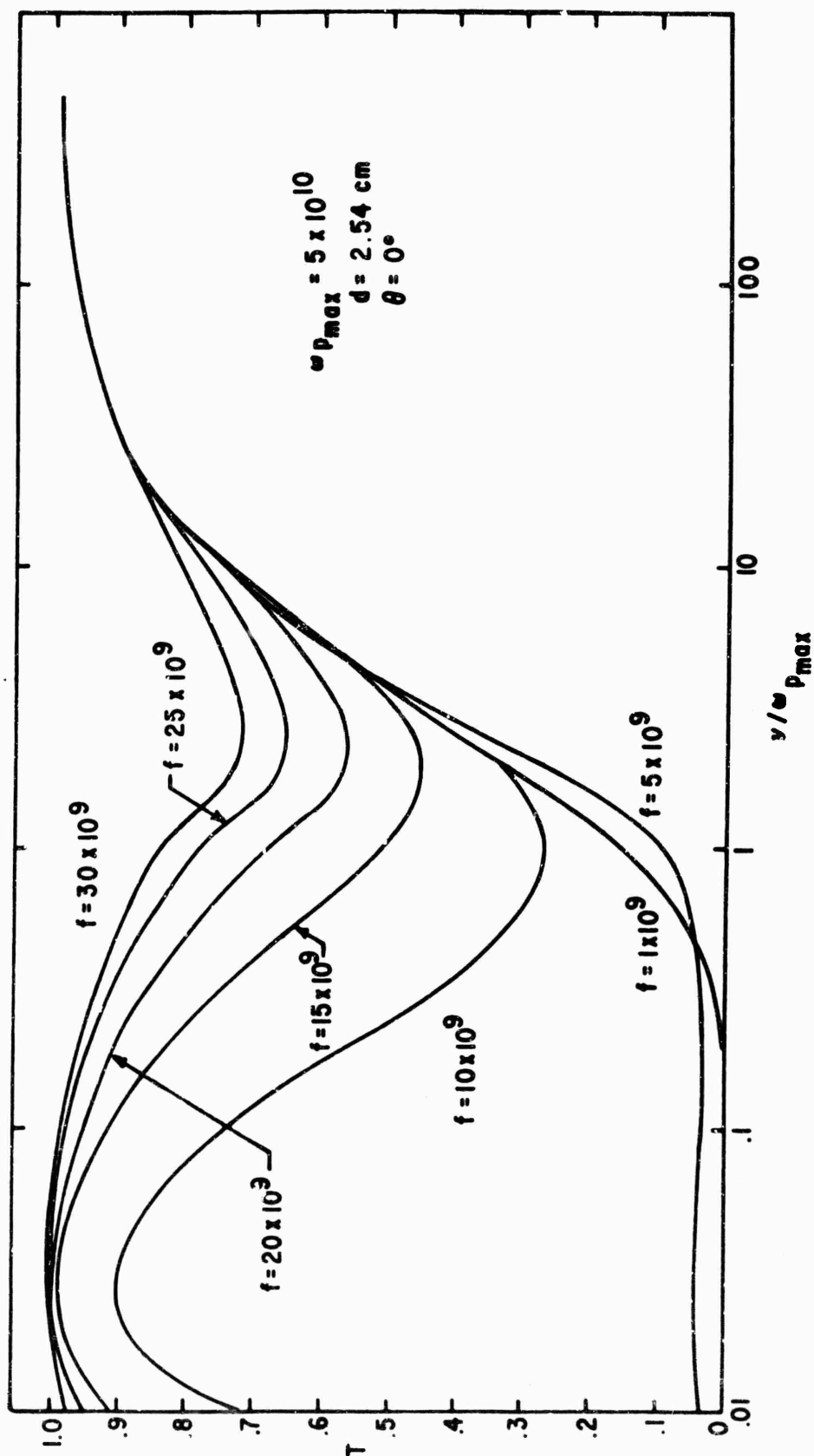
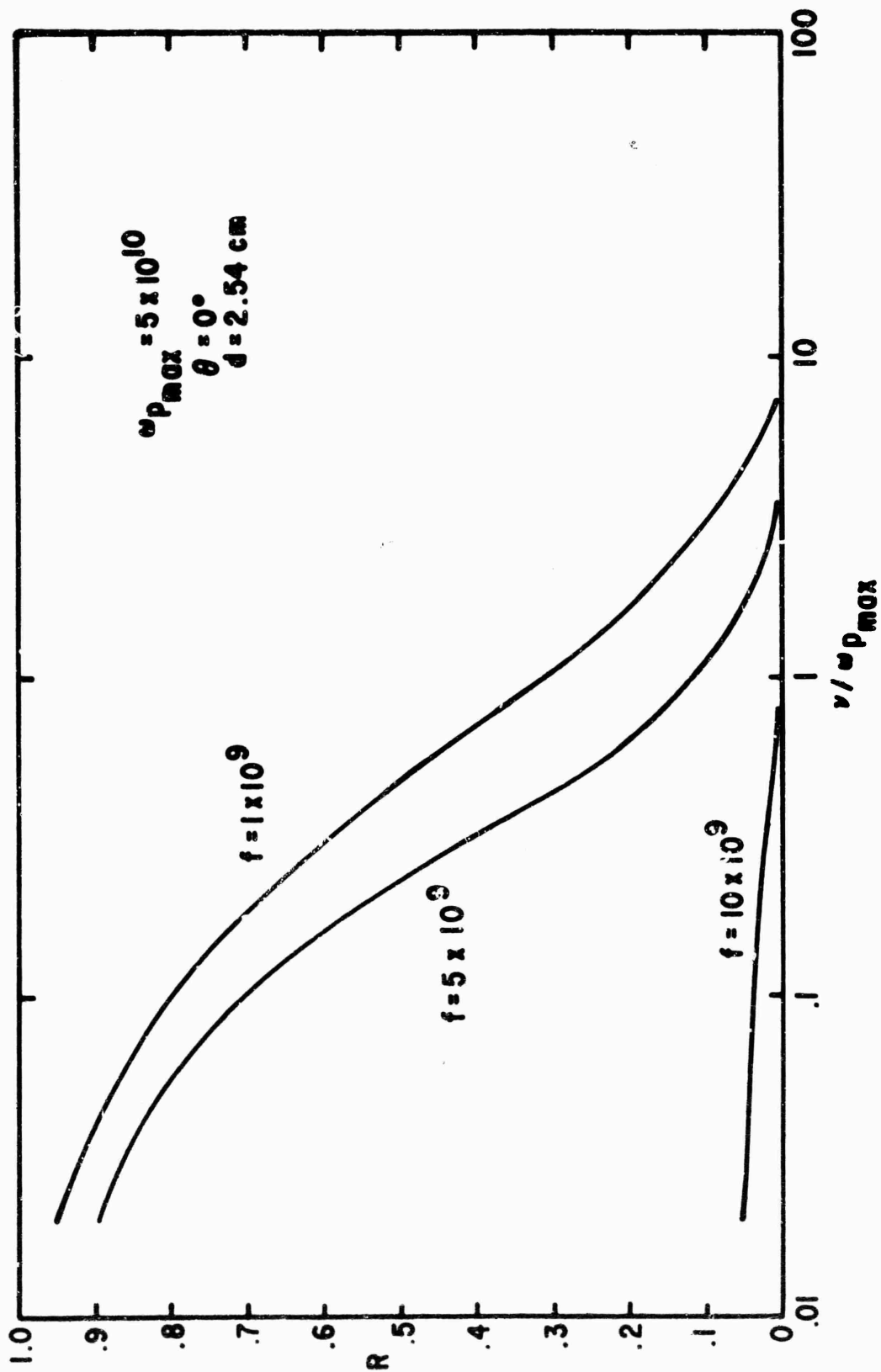Figure 6. Transmission Coefficient versus $\nu/\omega_{p_{max}}$

Figure 7. Reflection Coefficient versus $\nu/\omega_{p_{max}}$
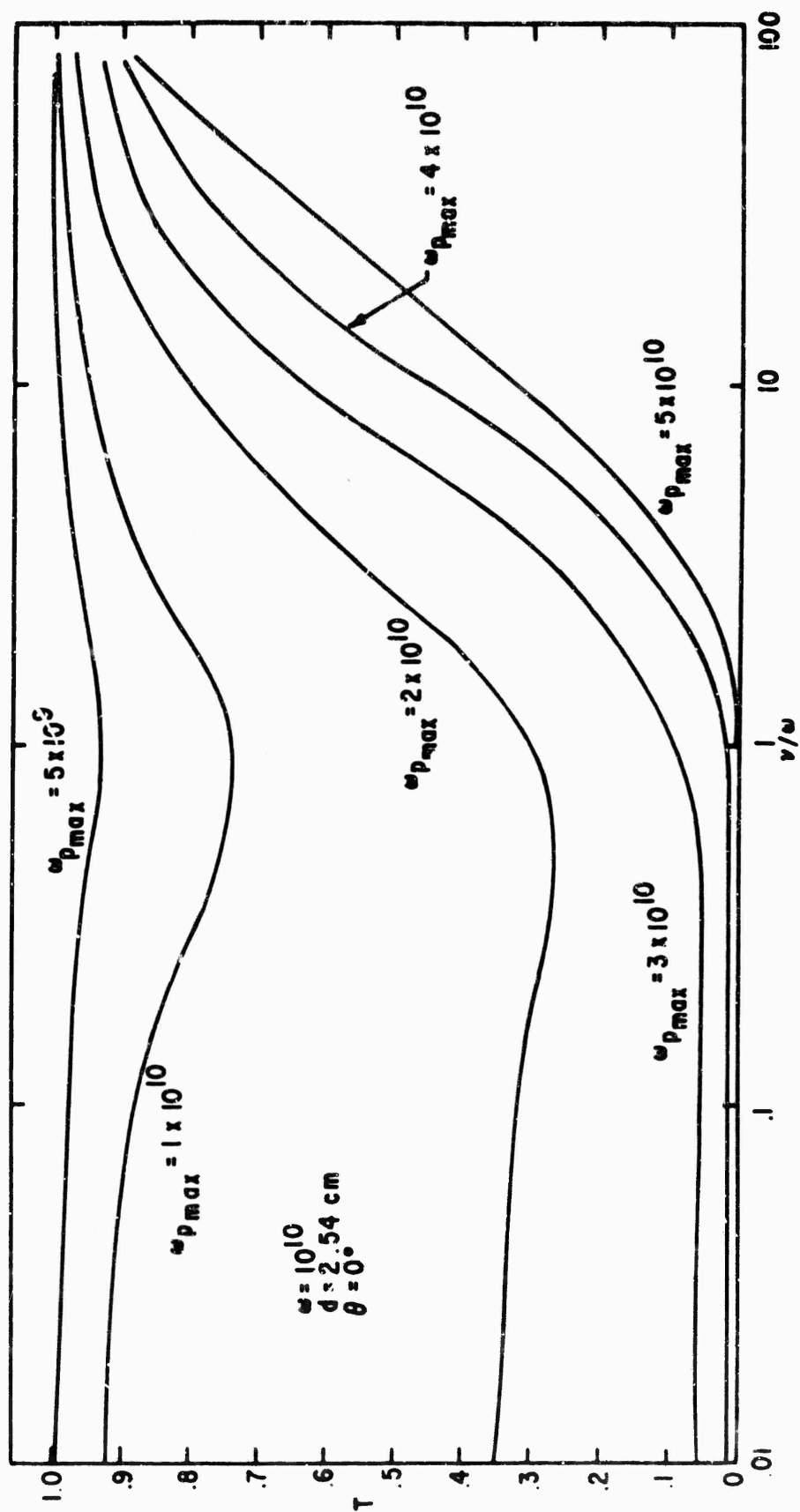
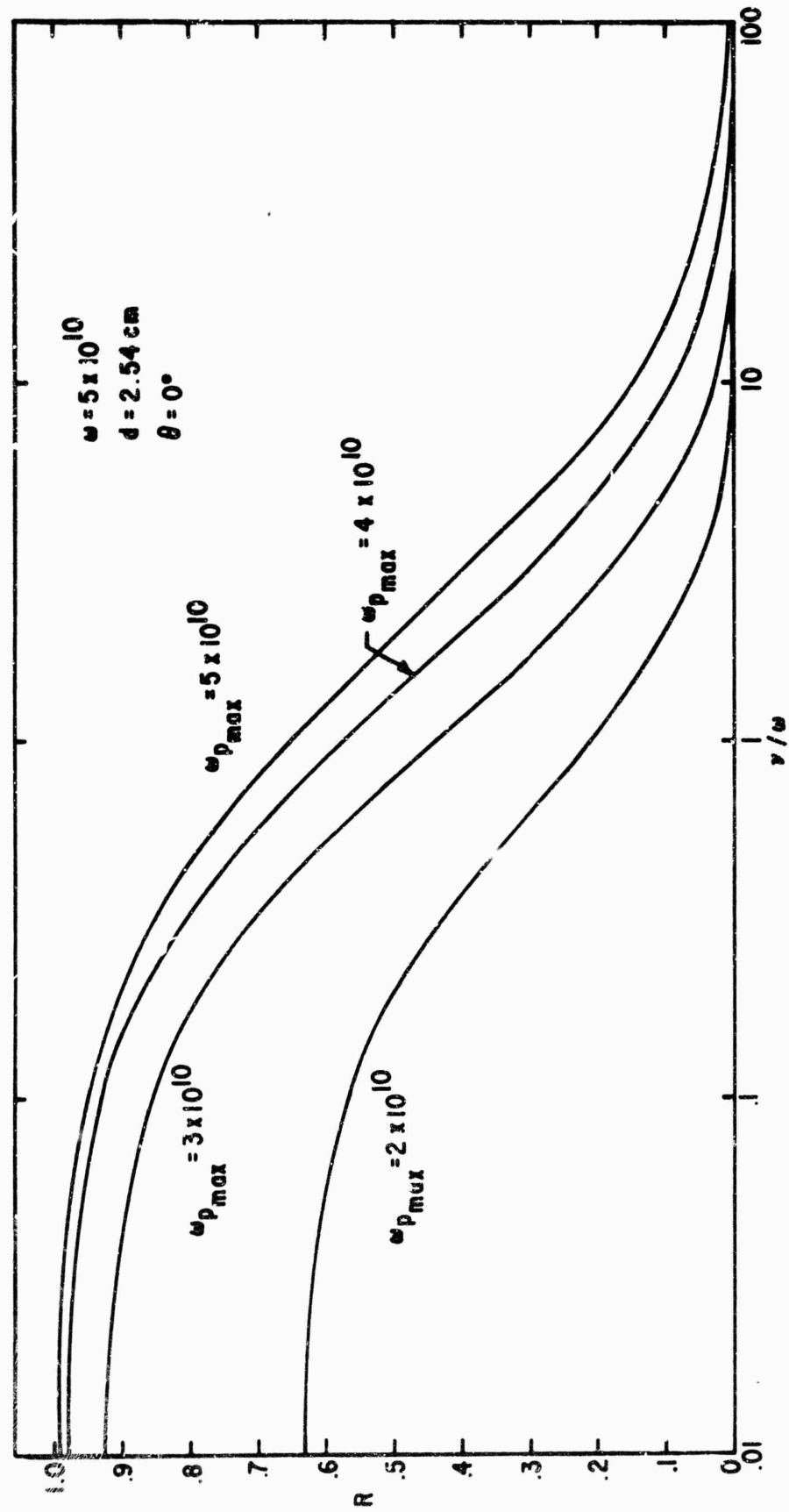Figure 8. Transmission Coefficient versus ν/ω

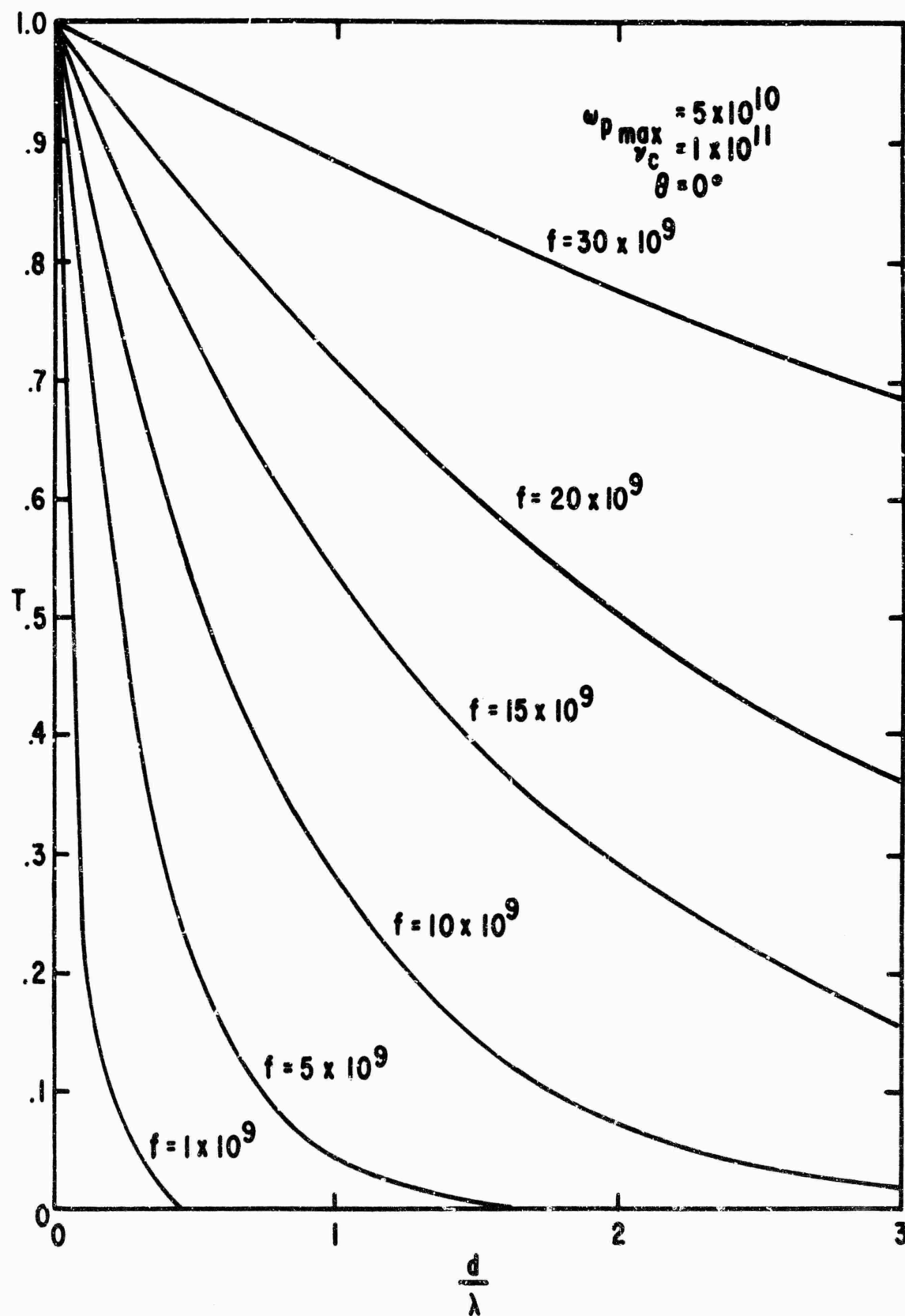Figure 9. Reflection Coefficient versus ν/ω
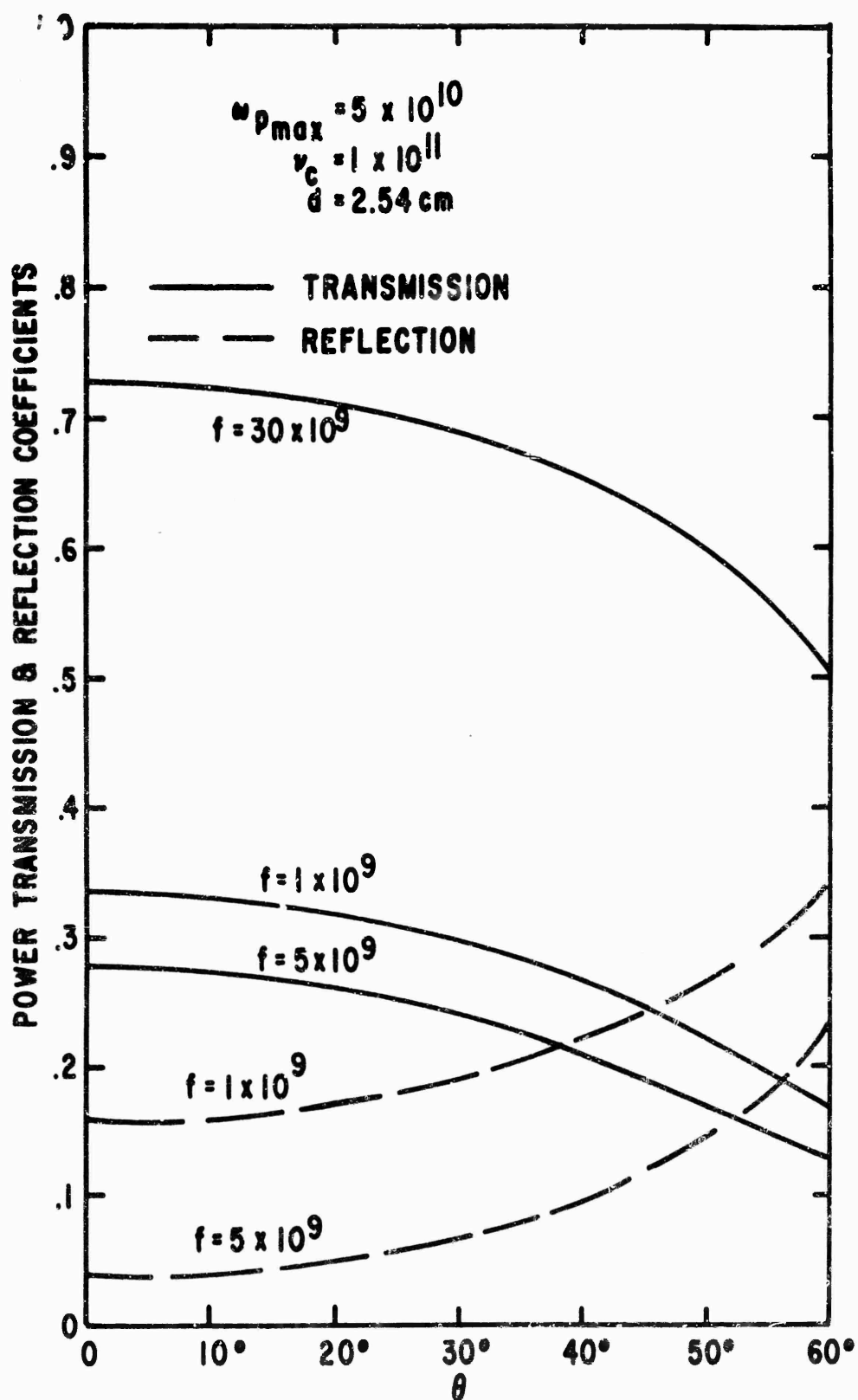
Figure 10. Transmission Coefficient versus d/λ

Figure 11. Transmission and Reflection Coefficients versus Angle of Incidence

APPENDIX I

SOLUTION TO THE WAVE EQUATION USING SEPARATION OF VARIABLES

The wave equation is

$$\frac{\partial^2 E_x(y,z)}{\partial y^2} + \frac{\partial^2 E_x(y,z)}{\partial y^2} + k_o^2 n^2(y) \; E_x(y,z) = 0 \tag{I-1}$$

Let

$$E_x(y,z) = F_{x1}(y) G_x(z) \tag{I-2}$$

Substituting into Equation (I-1) and dividing by $E_x(y,z)$ gives

$$\frac{1}{F_{x1}(y)} \frac{\partial^2 F_{x1}(y)}{\partial y^2} + \frac{1}{G_x(z)} \frac{\partial^2 G_x(z)}{\partial z^2} + k_o^2 n^2(y) = 0 \tag{I-3}$$

Rearranging Equation (I-3)

$$\frac{1}{F_{x1}(y)} \frac{\partial^2 F_{x1}(y)}{\partial y^2} + k_o^2 n^2(y) = -\frac{1}{G_x(y)} \frac{\partial^2 G_x(y)}{\partial z^2} \tag{I-4}$$

Let

$$\frac{1}{G_x(y)} \frac{d^2 G_x(z)}{dz^2} = -k_z^2 = \text{a constant} \tag{I-5}$$

We get two independent differential equations

$$\frac{1}{G_x(z)} \frac{d^2 G_x(z)}{dy^2} + k_z^2 = 0 \tag{I-6}$$

$$\frac{1}{F_{x1}(y)} \frac{d^2 F_{x1}(y)}{dy^2} + k_o^2 n^2(y) - k_z^2 = 0 \tag{I-7}$$

The solution to Equation (I-6) is

$$G_x(z) = Ke^{\pm jk_z z} \tag{I-8}$$

From the geometry shown in figure 1, $k_z = -k_o \sin\theta$.

If the condition is made that the wave propagates in the negative z-direction, the positive sign of the exponent can be dropped (reference 7). The solution to Equation (I-7) can now be written as

$$G_x(z) = Ke^{jk_o z \sin\theta} \tag{I-9}$$

Substituting the solution for $G_x(z)$ of Equation (I-9) into Equation (I-2) will give

$$E_x(y,z) = F_x(y)e^{jk_o z \sin\theta} \tag{I-10}$$

APPENDIX II

TRANSFORMATION INTO A CONFLUENT HYPERGEOMETRIC EQUATION

Equation (23) is

$$\frac{d^2 F_x(y)}{dy^2} + [A + B\eta^2] \, F_x(y) = 0 \tag{II-1}$$

Let

$$F_x(y) = e^{-\delta/2} \, v(\delta)$$

and

$$\delta = j\eta^2 \, \sqrt{B}$$

Using the chain rule for differentiation

$$\frac{dF_x}{dy} = \frac{dF_x}{d\delta} \frac{d\delta}{dy} \tag{II-2}$$

$$\frac{dF_x}{d\delta} = e^{-\delta/2} \left[\frac{dv(\delta)}{d\delta} - \frac{1}{2} \, v(\delta) \right] \tag{II-3}$$

$$\frac{d\delta}{dy} = -j^{1/2} 2 \delta^{1/2} B^{1/4} \tag{II-4}$$

$$\frac{dF_x}{dy} = -2B^{1/4} j^{1/2} e^{-\delta/2} \left[\delta^{1/2} \frac{dv(\delta)}{d\delta} - \frac{\delta^{1/2}}{2} \, v(\delta) \right] \tag{II-5}$$

Once more applying the chain rule

$$\frac{d^2F_x}{dy^2} = \frac{d}{d\delta}\left(\frac{dF_x}{dy}\right)\frac{d\delta}{dy} \qquad (II-6)$$

$$\frac{d}{d\delta}\left(\frac{dF_x}{dy}\right) = -2B^{1/4}j^{1/2}\left\{\delta^{1/2}\frac{d^2v(\delta)}{d\delta^2} + \left(\frac{1}{2}\delta^{-1/2} - \delta^{1/2}\right)\frac{dv(\delta)}{d\delta}\right.$$

$$\left. + \frac{1}{4}\left(\delta^{1/2}-\delta^{-1/2}\right)v(\delta)\right\}e^{-\delta/2} \qquad (II-7)$$

$$\frac{d^2F_x}{dy^2} = 4B^{1/2}j\left\{\delta\frac{d^2v(\delta)}{d\delta^2} + \left(\frac{1}{2} - \delta\right)\frac{dv(\delta)}{d\delta} + \frac{1}{4}(\delta-1)v(\delta)\right\}e^{-\delta/2} \qquad (II-8)$$

Substituting Equation (II-8) and the expression for F_x into Equation (23) and noting that $\beta\eta^2 = -j\sqrt{B}\,\delta$, gives

$$\frac{\delta d^2v(\delta)}{d\delta^2} + \left(\frac{1}{2} - \delta\right)\frac{dv(\delta)}{d\delta} - \left(\frac{\sqrt{B} + jA}{4\sqrt{B}}\right)v(\delta) = 0 \qquad (II-9)$$

Now, let

$$h = \frac{\sqrt{B} + jA}{2\sqrt{B}}$$

and Equation (II-9) becomes

$$\frac{\delta d^2v(\delta)}{d\delta^2} + \left(\frac{1}{2} - \delta\right)\frac{dv(\delta)}{d\delta} - \frac{h}{2}v(\delta) = 0 \qquad (II-10)$$

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The power transmission and reflection coefficients are derived for electromagnetic waves propagating throug a plasma slab having a parabolic electron distribution. The analysis considers transverse electromagnetic waves (TE Mode) impinging on a plasma slab at arbitrary angles of incidence. The solutions are in terms of complex hypergeometric functions and their derivatives. Numerical results for the transmission and reflection coefficients are plotted as functions of peak plasma frequency, peak collision frequency, signal frequency, slab thickness, and angle of incidence. The results of this study can be applied to transmission of electromagnetic energy through laboratory plasmas that are bounded by walls. Numerical results are in agreement with experimental results for a rectangular glow discharge plasma.

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